Lesson 11: Conditions on Measurements That Determine a Triangle

Student Outcomes

- Students understand that three given lengths determine a triangle, provided the largest length is less than the sum of the other two lengths; otherwise, no triangle can be formed.
- Students understand that if two side lengths of a triangle are given, then the third side length must be between the difference and the sum of the first two side lengths.
- Students understand that two angle measurements determine many triangles, provided the angle sum is less than $180°$; otherwise, no triangle can be formed.

Materials

Patty paper or parchment paper (in case dimensions of patty paper are too small)

Lesson Notes

Lesson 11 explores side-length requirements and angle requirements that determine a triangle. Students reason through three cases in the exploration regarding side-length requirements and conclude that any two side lengths must sum to be greater than the third side length. In the exploration regarding angle requirements, students observe the resulting figures in three cases and conclude that the angle sum of two angles in a triangle must be less than $180°$. Additionally, they observe that three angle measurements do not determine a unique triangle and that it is possible to draw scale drawings of a triangle with given angle measurements. Students are able to articulate the result of each case in the explorations.

Scaffolding:

An alternate activity that may increase accessibility is providing students with pieces of dry pasta (or other manipulatives) and rulers and then giving the task: "Build as many triangles as you can. Record the lengths of the three sides." Once students have constructed many triangles, ask, "What do you notice about the lengths of the sides?" Allow written and spoken responses. If necessary, ask students to construct triangles with particular dimensions (such as 3 cm, 4 cm, 1 cm; 3 cm, 4 cm, 5 cm; 3 cm, 4 cm, 8 cm) to further illustrate the concept.

Classwork

Exploratory Challenge 1 (8 minutes)

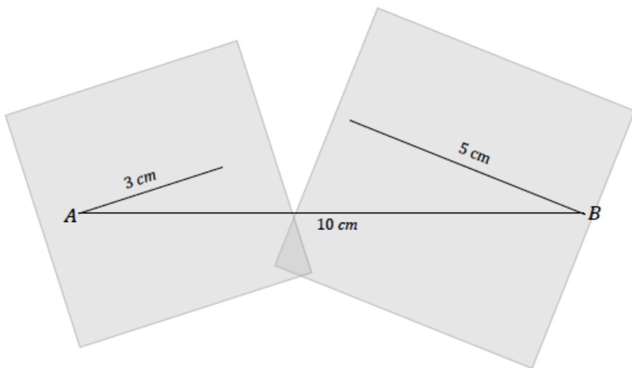
In pairs, students explore the length requirements to form a triangle.

Exploratory Challenge 1

a. **Can any three side lengths form a triangle? Why or why not?**

Possible response: Yes, because a triangle is made up of three side lengths; therefore, any three sides can be put together to form a triangle.

b. Draw a triangle according to these instructions:
- ✓ Draw segment AB of length 10 cm in your notebook.
- ✓ Draw segment BC of length 5 cm on one piece of patty paper.
- ✓ Draw segment AC of length 3 cm on the other piece of patty paper.
- ✓ Line up the appropriate endpoint on each piece of patty paper with the matching endpoint on segment AB.
- ✓ Use your pencil point to hold each patty paper in place, and adjust the paper to form $\triangle ABC$.



> *Scaffolding:*
>
> Allow students who feel comfortable with a compass to use one instead of the patty paper. The compass is adjusted to 5 cm (AC), and a circle is drawn with a center at A; similar steps are done for side $\overline{AB}$.

c. What do you notice?

$\triangle ABC$ cannot be formed because $\overline{AC}$ and $\overline{BC}$ do not meet.

d. What must be true about the sum of the lengths of $\overline{AC}$ and $\overline{BC}$ if the two segments were to just meet? Use your patty paper to verify your answer.

For $\overline{AC}$ and $\overline{BC}$ to just meet, the sum of their lengths must be equal to 10 cm.

e. Based on your conclusion for part (d), what if $AC = 3$ cm as you originally had, but $BC = 10$ cm. Could you form $\triangle ABC$?

$\triangle ABC$ can be formed because $\overline{AC}$ and $\overline{BC}$ can meet at an angle and still be anchored at A and B.

f. What must be true about the sum of the lengths of $\overline{AC}$ and $\overline{BC}$ if the two segments were to meet and form a triangle?

For $\overline{AC}$ and $\overline{BC}$ to just meet and form a triangle, the sum of their lengths must be greater than 10 cm.

Discussion (7 minutes)

MP.7

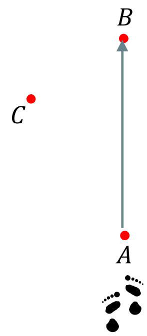
- Were you able to form $\triangle ABC$? Why or why not? Did the exercise confirm your prediction?
  - *We could not form $\triangle ABC$ because sides $\overline{BC}$ and $\overline{AC}$ are too short to meet.*
- What would the sum of the lengths of $\overline{BC}$ and $\overline{AC}$ have to be to just meet? Describe one possible set of lengths for $\overline{BC}$ and $\overline{AC}$. Would these lengths form $\triangle ABC$? Explain why or why not.
  - *The lengths would have to sum to 10 cm to just meet (e.g., $AC = 3$ cm and $BC = 7$ cm). Because the segments are anchored to either endpoint of $\overline{AB}$, the segments form a straight line or coincide with $\overline{AB}$. Therefore, $\triangle ABC$ cannot be formed since A, B, and C are collinear.*

MP.7

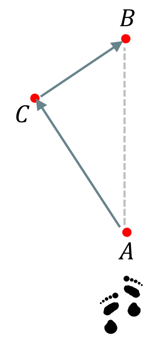
- If a triangle cannot be formed when the two smaller segments are too short to meet or just meet, what must be true about the sum of the lengths of the two smaller segments compared to the longest length? Explain your answer using possible measurements.
  - *The sum of the two smaller lengths must be greater than the longest length so that the vertices will not be collinear; if the sum of the two smaller lengths is greater than the longest length while anchored at either endpoint of $\overline{AB}$, the only way they can meet is if A, B, and C are not collinear. One possible set of lengths is $AC = 4$ cm and $BC = 7$ cm.*

Help students recognize this fundamental inequality by illustrating it with an image of walking between two points, A and B.

Observe the two pathways to get from A to B. Pathway 1 is a straight path from A to B. Pathway 2 requires you to walk through a point that does not lie on the straight path. Clearly, the total distance when walking through C is greater than the distance of walking the straight path. This idea can be visualized from A, B, or C. Hence, the length of any one side of a triangle is always less than the sum of the lengths of the remaining sides.



Pathway 1



Pathway 2

MP.7

- Given two side lengths of a triangle, the third side length must be between the difference of the two sides and the sum of the two sides. For example, if a triangle has two sides with lengths 2 cm and 5 cm, then the third side length must be between the difference ($5 \text{ cm} - 2 \text{ cm}$) and the sum ($2 \text{ cm} + 5 \text{ cm}$). Explanation: Let x be the length of the third side in centimeters. If $x \leq 5$, then the largest side length is 5 cm, and $2 + x > 5$, or $x > 5 - 2$. If $x \geq 5$, then x cm is the longest side length and $5 + 2 > x$. So, $5 - 2 < x < 2 + 5$.

Exercise 1 (4 minutes)

In this exercise, students must consider the length of the last side from two perspectives: one where the last side is not the longest side and one where the last side is the longest side. With these two considerations, the third side is a range of lengths, all of which satisfy the condition that the longest side length is less than the sum of the other two side lengths.

Exercise 1

Two sides of $\triangle DEF$ have lengths of 5 cm and 8 cm. What are all the possible whole number lengths for the remaining side?

The possible whole-number side lengths in centimeters are 4, 5, 6, 7, 8, 9, 10, 11, and 12.

Exploratory Challenge 2 (8 minutes)

Students explore the angle measurement requirements to form a triangle in pairs. Encourage students to document their exploration carefully, even if their results are not what they expected.

Exploratory Challenge 2

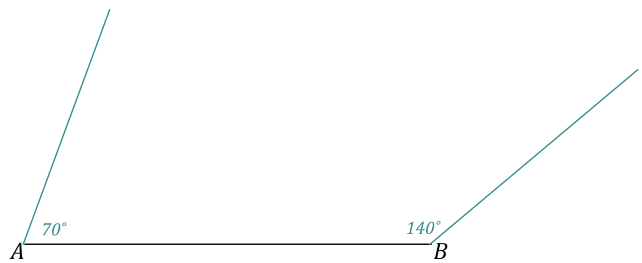
a. **Which of the following conditions determine a triangle? Follow the instructions to try to draw $\triangle ABC$. Segment AB has been drawn for you as a starting point in each case.**

i. **Choose measurements of $\angle A$ and $\angle B$ for $\triangle ABC$ so that the sum of measurements is greater than $180°$. Label your diagram.**

Your chosen angle measurements: $\angle A = 70°$ $\angle B = 140°$

Were you able to form a triangle? Why or why not?

Selected angle measurements and the corresponding diagram indicate one possible response.



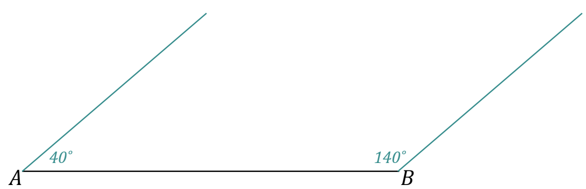
We were not able to form a triangle because the non-horizontal ray of $\angle A$ and the non-horizontal ray of $\angle B$ do not intersect.

ii. **Choose measurements of $\angle A$ and $\angle B$ for $\triangle ABC$ so that the measurement of $\angle A$ is supplementary to the measurement of $\angle B$. Label your diagram.**

Your chosen angle measurements: $\angle A = 40°$ $\angle B = 140°$

Were you able to form a triangle? Why or why not?

Selected angle measurements and the corresponding diagram indicate one possible response.



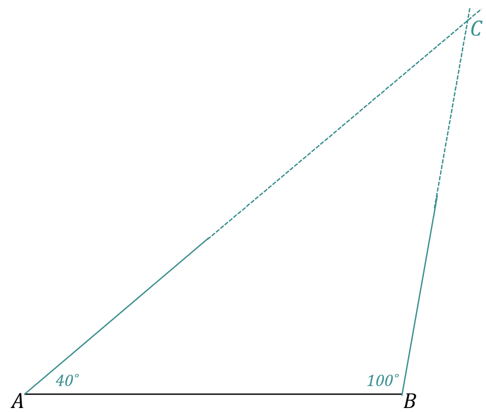
We were not able to form a triangle because the non-horizontal ray of $\angle A$ and the non-horizontal ray of $\angle B$ do not intersect; the non-horizontal rays look parallel.

iii. **Choose measurements of $\angle A$ and $\angle B$ for $\triangle ABC$ so that the sum of measurements is less than $180°$. Label your diagram.**

Your chosen angle measurements: $\angle A = 40°$ $\angle B = 100°$

Were you able to form a triangle? Why or why not?

Angle measurements and the corresponding diagram indicate one possible response.



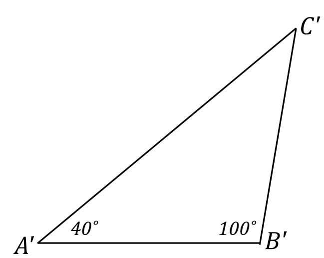
We were able to form a triangle because the non-horizontal ray of $\angle A$ and the non-horizontal ray of $\angle B$ intersect.

> *Scaffolding:*
>
> Teachers may want to demonstrate the informal proof and illustrate that the angle sum of a triangle is $180°$. Do this by tearing off the corners of a triangle and forming a straight line by placing the three angles adjacent to each other. Note that this is an extension and is not formally discussed until Grade 8.

b. **Which condition must be true regarding angle measurements in order to determine a triangle?**

The sum of two angle measurements of a triangle must be less than $180°$.

c. **Measure and label the formed triangle in part (a) with all three side lengths and the angle measurement for $\angle C$. Now, use a protractor, ruler, and compass to draw $\triangle A'B'C'$ with the same angle measurements but side lengths that are half as long.**



Students should begin by drawing any one side length at a length half as much as the corresponding side in $\triangle ABC$ and then drawing angles at each end of this line segment. Students should recognize that $\triangle A'B'C'$ is a scale drawing of $\triangle ABC$. Ask students to mark all length measurements as a means of verifying that they are indeed half as long as the corresponding sides of the original triangle.

d. **Do the three angle measurements of a triangle determine a unique triangle? Why or why not?**

Three angles do not determine a unique triangle. For a given triangle with three provided angle measurements, another triangle can be drawn with the same angle measurements but with side lengths proportional to those side lengths of the original triangle.

Discussion (7 minutes)

- Why couldn't $\triangle ABC$ be formed in case (i), when the sum of the measurements of $\angle A$ and $\angle B$ were greater than $180°$?
  - *The non-horizontal rays will not intersect due to the angle they form with $\overline{AB}$.*
- Why couldn't $\triangle ABC$ be formed in case (ii), when the sum of the measurements of $\angle A$ and $\angle B$ were supplementary?
  - *The non-horizontal rays will not intersect. The lines look parallel; if they are extended, they seem to be the same distance apart from each other at any given point.*

Confirm that the two non-horizontal rays in this case are, in fact, parallel and two supplementary angle measurements in position to be two angles of a triangle will always yield parallel lines.

- What conclusion can we draw about any two angle measurements of a triangle, with respect to determining a triangle?
  - *The sum of any two angles of a triangle must be less than $180°$ in order to form the triangle.*
- Do the three angle measurements of a triangle guarantee a unique triangle?
  - *No, we drew a triangle that had the same angle measurements as our triangle in case (iii) but with side lengths that were half the length of the original triangle*.

Remind students of their work with scale drawings. Triangles that are enlargements or reductions of an original triangle all have equal corresponding angle measurements but have side lengths that are proportional.

Exercise 2 (4 minutes)

Exercise 2

Which of the following sets of angle measurements determines a triangle?

a. $30°, 120°$ ***Determines a triangle***

b. $125°, 55°$ ***Does not determine a triangle***

c. $105°, 80°$ ***Does not determine a triangle***

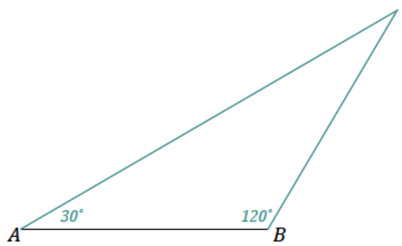
d. $90°, 89°$ ***Determines a triangle***

e. $91°, 89°$ ***Does not determine a triangle***

Choose one example from above that does determine a triangle and one that does not. For each, explain why it does or does not determine a triangle using words and a diagram.

Possible response:

The angle measurements in part (a) determine a triangle because the non-horizontal rays of the $30°$ *angle and the* $120°$ *angle will intersect to form a triangle.*



The angle measurements in part (c) do not determine a triangle because the non-horizontal rays of the $105°$ *angle and the* $80°$ *angle will not intersect to form a triangle.*

Closing (2 minutes)

Lesson Summary

- **Three lengths determine a triangle provided the largest length is less than the sum of the other two lengths.**
- **Two angle measurements determine a triangle provided the sum of the two angle measurements is less than $180°$.**
- **Three given angle measurements do not determine a unique triangle.**
- **Scale drawings of a triangle have equal corresponding angle measurements, but corresponding side lengths are proportional.**

Exit Ticket (5 minutes)

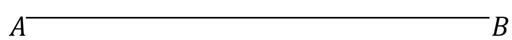
Name ____ Date ____________________

Lesson 11: Conditions on Measurements That Determine a Triangle

Exit Ticket

1. What is the maximum and minimum whole number side length for $\triangle XYZ$ with given side lengths of 3 cm and 5 cm? Please explain why.

2. Jill has not yet studied the angle measurement requirements to form a triangle. She begins to draw side $\overline{AB}$ of $\triangle ABC$ and considers the following angle measurements for $\angle A$ and $\angle B$. Describe the drawing that results from each set.



   a. $45°$ and $135°$

   b. $45°$ and $45°$

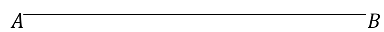
   c. $45°$ and $145°$

Exit Ticket Sample Solutions

1. What is the minimum and maximum whole number side length for $\triangle XYZ$ with given side lengths of 3 cm and 5 cm? Please explain why.

   Minimum: 3 cm. Maximum: 7 cm. Values above this maximum and below this minimum will not satisfy the condition that the longest side length is less than the sum of the other two side lengths.

2. Jill has not yet studied the angle measurement requirements to form a triangle. She begins to draw side $\overline{AB}$ as a horizontal segment of $\triangle ABC$ and considers the following angle measurements for $\angle A$ and $\angle B$. Describe the non-horizontal rays in the drawing that results from each set.



   a. $45°$ and $135°$

      The non-horizontal rays of $\angle A$ and $\angle B$ will not intersect to form a triangle; the rays will be parallel to each other.

   b. $45°$ and $45°$

      The non-horizontal rays of $\angle A$ and $\angle B$ will intersect to form a triangle.

   c. $45°$ and $145°$

      The non-horizontal rays of $\angle A$ and $\angle B$ will not intersect to form a triangle.

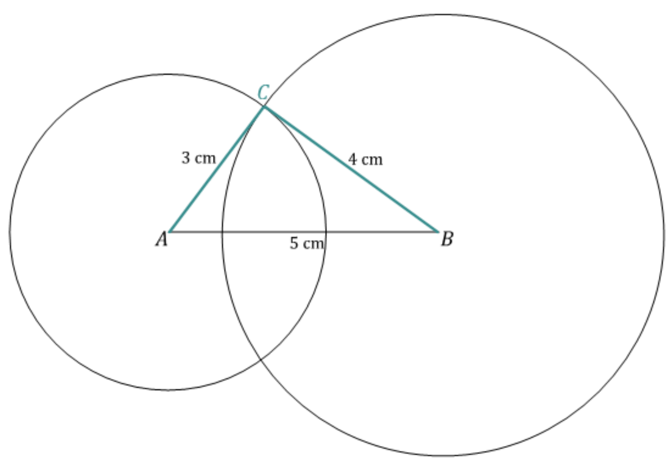
Problem Set Sample Solutions

Scaffolding:

Lessons 7 and 8 demonstrate how to use a compass for questions such as Problem 1.

1. Decide whether each set of three given lengths determines a triangle. For any set of lengths that does determine a triangle, use a ruler and compass to draw the triangle. Label all side lengths. For sets of lengths that do not determine a triangle, write "Does not determine a triangle," and justify your response.

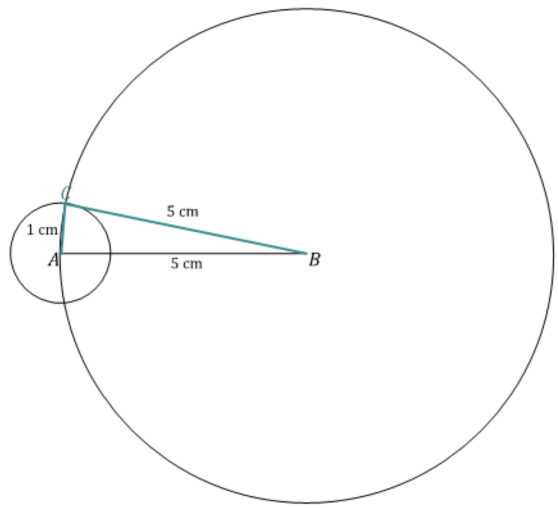
   a. 3 cm, 4 cm, 5 cm



b. **1 cm, 4 cm, 5 cm**

Does not determine a triangle. The shorter lengths are too short to form a triangle. They will only form a segment equal to the length of the longest side.

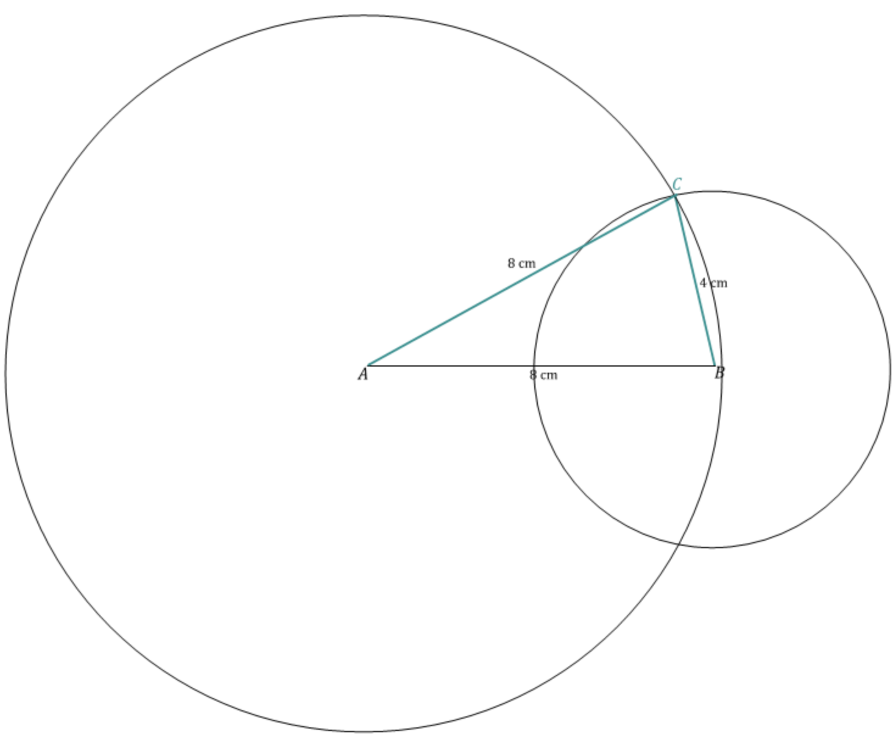
c. **1 cm, 5 cm, 5 cm**



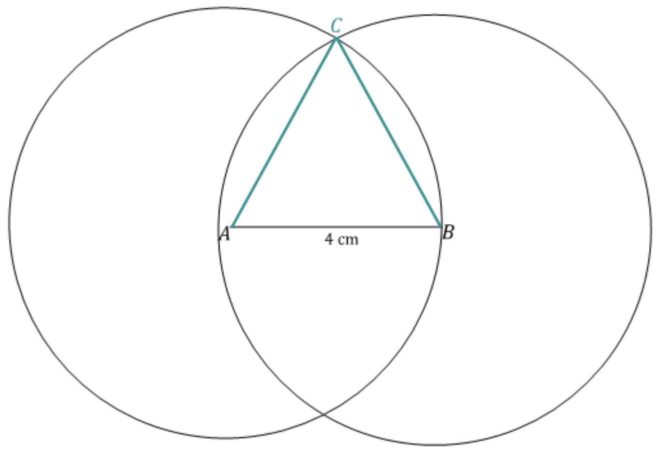
d. **8 cm, 3 cm, 4 cm**

Does not determine a triangle. The shorter lengths are too short to form a triangle. They will only form a segment that is shorter than the length of the longest side.

e. **8 cm, 8 cm, 4 cm**



f. 4 cm, 4 cm, 4 cm



2. For each angle measurement below, provide one angle measurement that will determine a triangle and one that will not determine a triangle. Provide a brief justification for the angle measurements that will not form a triangle. Assume that the angles are being drawn to a horizontal segment AB; describe the position of the non-horizontal rays of angles $\angle A$ and $\angle B$.

$\angle A$	$\angle B$: A Measurement That Determines a Triangle	$\angle B$: A Measurement That *Does Not* Determine a Triangle	Justification for No Triangle
$40°$	*One possible answer:* $30°$	*One possible answer:* $150°$	*The non-horizontal rays do not intersect.*
$100°$	*One possible answer:* $30°$	*One possible answer:* $150°$	*The non-horizontal rays do not intersect.*
$90°$	*One possible answer:* $30°$	*One possible answer:* $90°$	*The non-horizontal rays do not intersect.*
$135°$	*One possible answer:* $30°$	*One possible answer:* $80°$	*The non-horizontal rays do not intersect.*

Note:

- Measurements that determine a triangle should be less than $180° - (\text{the measurement of } \angle A)$.
- Measurements that do not determine a triangle should be greater than $180° - (\text{the measurement of } \angle A)$.

3. For the given side lengths, provide the minimum and maximum whole number side lengths that determine a triangle.

Given Side Lengths	Minimum Whole Number Third Side Length	Maximum Whole Number Third Side Length
5 cm, 6 cm	2 cm	10 cm
3 cm, 7 cm	5 cm	9 cm
4 cm, 10 cm	7 cm	13 cm
1 cm, 12 cm	12 cm	12 cm